

Issues of Commonality

The completed image connects the start of the first line with the end of the last line using a simple counting rule, P . As such, there are two ways to draw each static image. Additionally, the number of lines required, and how subsets of those lines relate to one another depend on commonality between the parameters. Before examining commonality, it is useful to describe distinct images.

Distinct Images. An ESA image has one line in and one line out of the top of the image. The values of n , S , P , and J determine how this first line (shown in red in the previous section), and hence the image, is drawn. Since either line can be the first drawn, there are two ways to draw any image.

The figure to the right shows all distinct images given $n = 5$ and $S = 3$ for $1 \leq P < 15 = n \cdot S$. The left images are $J = 1$ or 4, and the right are $J = 2$ or 3. There are 10 distinct ESA images given these values of n and S (and thus $1 \leq J < n$).

We see that one may restrict $J < n/2$ and $P < n \cdot S/2$ because larger values of J and P only alter the direction that the image is drawn. Of course, there will be times when we prefer to visualize the way that the image is drawn, and in that instance, we will resort to $J > n/2$ or $P > n \cdot S/2$ to accomplish this task.

We also see that if P is a factor of S , then the image is simply the VF. Although note that if $P = 1$ or 14, the VF is drawn in 15 connected segments, but if $P = 3$ or 12, the VF is drawn in 5.

Counting lines in the Image. The table makes clear that it takes 15 lines to create the images in Row B, but visual inspection suggests that each image has only 10 line segments. The reason for this distinction has to do with the basic nature of ESA images: they must be *continuously-drawn*. Both images are created with 15 segments connecting 0-2-4-6-8-10-12-14-1-3-5-7-9-11-13-0. Notice that it takes two passes around the 15 subdivision endpoints to complete the circuit and finish the image. In this case, even endpoints are used in the first half and odd endpoints are used in the second half, with the line from 14-1 bridging the even and odd segments.

Row A is an even simpler example of why the number of lines in an image may be deceiving. It looks like each image was created in 5 lines, but each of those lines involves 3 connected collinear segments given $P = 1$ (or 14), but only 1 line per side when $P = 3$ (or 12). The difference is that the SCF = 1 in the first instance and 3 in the second. Before defining SCF, we need to define VCF.

Vertex Common Factor, VCF. The first form of commonality is between the number of vertices, n , and the number of jumps between vertices, J ; they are used to create the VF. This is called the *vertex common factor*,

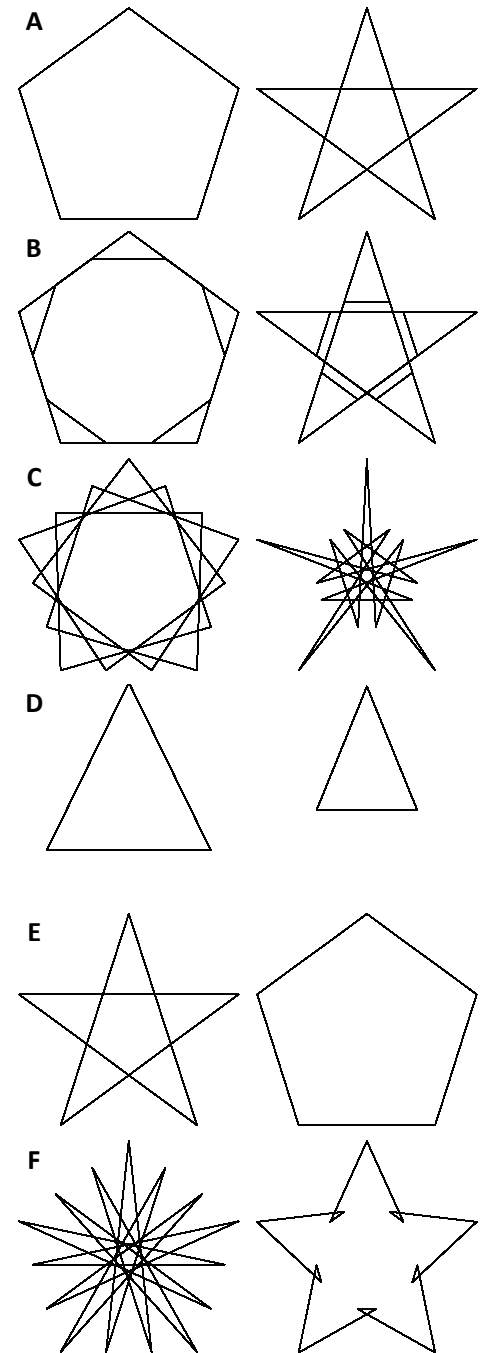
$$\text{VCF} = \text{GCD}(n, J) \text{ where GCD is the greatest common divisor function.}$$

The number of used vertices and lines in the VF is:

$$v_{\text{used}} = n/\text{VCF}.$$

The simplest example of $\text{VCF} > 1$ is a 6,2-star whose VF is an equilateral triangle because the circuit is complete once the even vertices are connected to one another. *NOTE: A 6,2-star with 6-points cannot be continuously drawn.*

One can always obtain the same image by dividing both n and J by the VCF (in the single jump mode), but in the jump set mode we discuss later in the chapter, this is no longer the case. On the other hand, it is often useful to maintain the notation of n , rather than use $n' = n/\text{VCF}$ and $J' = J/\text{VCF}$, so that we can easily see commonality between n and J in the resulting image.



Row	P	$nS-P$	Lines in		
			Cycle	Total	SCF
A	1	14	3	15	1
B	2	13	3	15	1
A	3	12	1	5	3
C	4	11	3	15	1
D	5	10	3	3	5
E	6	9	1	5	3
F	7	8	3	15	1

We take the most readily understood example (because we live in a world with clocks), so $n = 12$. The VFs for the resulting images are a dodecagon, hexagon, square, triangle, 12,5-star, and a line segment as $J = 1, 2, 3, 4, 5$, and 6 due to the VCF values of 1, 2, 3, 4, 1, and 6 corresponding to these values of J . $VCF > 1$ is the reason that a 12,5-star is the only distinct **ESA** star with 12-points. *A 12,2-star can be drawn by creating two hexagons, one connecting even vertices and the other connecting odd vertices, but it cannot be continuously-drawn because there is no way to venture from even vertices to odd vertices given n is even and $J = 2$.*

By contrast, it is ALWAYS possible to obtain an $n,2$ -star for odd $n = 2k+1 > 3$ because the k^{th} line ends at vertex $2k = n-1$. Moreover, the next line is a jump of 2 that ends at vertex 1, after which all odd vertices are connected in k more jumps of two, ending at vertex $n = 2k+1$ which is also the top of the image, or vertex 0. (Note that the order of vertices used to create a 15,2 star are the same as the subdivision endpoints used to create the images in Row B discussed above. The first 7 connect even vertices, the 8th connects even to odd (14-1), and the last 7 connect odd vertices.)

Subdivision Common Factor, SCF. Given n/VCF used vertices, n/VCF lines in the VF, and S subdivisions per VF line, there are $S \cdot n/VCF$ possible subdivision endpoints that can be used in the image. The second form of commonality, the *subdivision common factor*, SCF, is:

$$SCF = \text{GCD}(P, S \cdot n/VCF).$$

An image is *full density* if $SCF = 1$, [E5.4.2](#). All its subdivision endpoints are used. The *Sequence Player* mode, discussed in Chapter 12, shows full density images exclusively. In general, the number of used endpoints (and lines) in the image is:

$$\text{Lines} = S \cdot n/VCF/SCF.$$

Lines and SCF values are shown in the table beneath the images. In Row D, both images are isosceles triangles connecting subdivision endpoints 5, 10, and 0 since $SCF = 5$. $P = 3$ creates Row A images connecting subdivision endpoints 3, 6, 9, 12, and 0 and result in the VF. By contrast, $P = 6$ in Row E connects endpoints 6, 12, 3, 9, and 0 so a pentagram results if $J = 1$; and a pentagon results if $J = 2$ (but note that this pentagon is drawn counterclockwise since subdivision endpoint 6 is vertex 4 of the 5-gon given $J = 2$). In both rows, this happens because $SCF = 3$.

Vertical Symmetry. Any image will have a vertical line of symmetry because there are two ways to draw the image (by switching which is the first line and which is the last line drawn in the image). The order of drawing is switched by changing either J of P . For example, suppose $VCF = 1$ and $J_0 < n_0$, $P_0 < n_0 \cdot S_0$. Then,

$$(n_0, S_0, P_0, J_0), (n_0, S_0, P_0, n_0 - J_0), (n_0, S_0, n_0 \cdot S_0 - P_0, J_0), \text{ and } (n_0, S_0, n_0 \cdot S_0 - P_0, n_0 - J_0) \text{ all produce the same image.}$$

The middle two are drawn one way and the first and last are drawn the other, [E6.2](#).

Rotational Symmetry. An image will have rotational symmetry equal to the number of polygonal vertices included in the final image. Mathematicians call this the *order of rotational symmetry*. Given this, to understand an image, one need only consider the part of the image that spans from the top of the image (the start of the first line) to the next time a polygonal vertex is encountered using counting rule P . For simplicity we call such a sub-image a *cycle*. [It is worth noting that it is mathematically more accurate to call this a *vertex-to-vertex cycle generator* because the entire image is called a cycle.]

Cycles. In Row B, given $P = 2$, the first cycle ends at vertex **2** for the left image and vertex **4** for the right image. Interestingly, in Row F, given $P = 7$, the first cycle also ends at vertex **2** for the left image and vertex **4** for the right image. In Row C, given $P = 4$, the first cycle ends at vertex **4** for the left image and vertex **3** for the right image. But in Row D, the first cycle ends at 0 for both (since there is only one cycle).

In general, the number of lines in a cycle, C , is:

$$C = S/\text{GCD}(S, P).$$

As a result, given $S = 3$, $P = 3, 6, 9$, and 12 all produce one-line cycles; it is worth noting that all other P have 3-line cycles. The number of cycles in the image, M , and hence the order of rotational symmetry, is given by:

$$M = \text{Order of Rotational Symmetry} = \text{Lines}/C.$$

By row from A to F, the order of rotational symmetry is therefore 5, 5, 5, 1, 5, 5. You can read more about cycles in [E5](#), more about various forms of symmetry in [E6](#), as well as symmetry across a cycle in [E7.4](#).